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Deformations on coadjoint orbits

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Abstract. *A deformation of the polynomial algebra $S(\mathcal{G})$ on $\mathcal{G}^*$ when $S(\mathcal{G})$ is a free $I(\mathcal{G})$ module ($I(\mathcal{G})$ = algebra of invariant polinomials). This deformation restricts nicely to a large class of orbits. We also give an example to show that deformations of $S(\mathcal{G})$ restricting to orbits may not always be defined by bidifferential operators.*

0. INTRODUCTION

Deformations (and * products which form a particular class of deformations) have been introduced by M. Flato, C. Fronsdal and A. Lichnerowicz (see for instance [1]) in order to give a formulation of quantum mechanics without operators in the general framework of a Poisson manifold; this geometrical approach to quantum theory is of a different nature from what one calls usually geometric quantization [2, 3].

Geometric quantization has been from the start deeply related to the method of orbits in the representation theory of Lie groups. In this context one of the important results has been the description of unitary irreducible representations of solvable groups [4] and of certains classes of representations of more general

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Lie groups using quantization of an orbit W of the group G in the dual $\mathcal{G}^*$ of its Lie algebra.

These considerations have motivated an attempt to construct representations of a Lie group G using a certain deformation of the usual product of functions on such an orbit W. This has been done by C. Fronsdal in certain semi simple and compact cases [5] and by one of us (D.A.) and J.C. Cortet in the nilpotent case [6].

Although this paper does not contain the word representation and is basically of a technical nature, it is concerned with general constructions of deformations on a set of orbits of a group G in $\mathcal{G}^*$ and is thus deeply related to the above mentioned program. More precisely our motivations are the following.

There exist two different constructions of deformations on a set of orbits of G: one, which is very algebraic in nature, and works for regular orbits of semi simple groups [7]; the other one, which is more analytic, and works for all orbits of a nilpotent Lie group [6]. Our first aim has been to incorporate these two constructions in a unified general context.

On the other hand the deformations constructed for orbits of a nilpotent group turn out to be «glued» together on a union of orbits which form a Zariski open set U in $\mathcal{G}^*$. This has allowed the construction of a so called nilpotent Fourier transform [8]. Our second aim has been to extend the deformation built on U to a deformation defined on the whole polynomial algebra on $\mathcal{G}^*$; this would in principle allow a «globalisation» of the nilpotent Fourier transform.

The results of this paper are the following. We first give a construction of a deformation of the polynomial algebra $S(\mathcal{G})$ on $\mathcal{G}^*$ in the case where $S(\mathcal{G})$ is a free $I(\mathcal{G})$ module ($I(\mathcal{G})$ is the algebra of invariant polynomials). This deformation restricts nicely to those orbits W which have the property that the only polynomials vanishing on W are elements of $I(\mathcal{G})$. This construction generalizes the one which is known in the semi simple case and contains a number of nilpotent situations. This goes in the direction of unification mentioned above.

We then give an example to show that one may not hope to have a $*$ product on $S(\mathcal{G})$, defined by bidifferential operators and restricting nicely to a large family of orbits in $\mathcal{G}^*$.

As deformations defined by bidifferential operators are easier to handle technically we end up by constructing a differential deformation for the class of special nilpotent algebras; this deformation restricts nicely to the orbits of G contained in a certain Zariski open set U of $\mathcal{G}^*$. This deformation is distinct from the one considered in [6].

It is clear that these results should now be applied to representation theory and in particular should be used in the construction of an adapted Fourier transform.

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1. DEFINITIONS AND NOTATIONS

Let G be a real connected Lie group; let $\mathscr{G}$ be the algebra of G and $\mathscr{G}^*$ the dual of $\mathscr{G}$. The group G acts on $\mathscr{G}^*$ by the coadjoint representation

$$G \times \mathscr{G}^* \to \mathscr{G}^* : (g, \xi) \to g \cdot \xi \underset{\text{def}}{=} \xi \circ \operatorname{Ad} g^{-1}.$$

If $X \in \mathscr{G}$ one denotes by X^*, the fundamental vector field on $\mathscr{G}^*$ associated this action:

$$X^*_\xi \underset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t} (\exp - tX \cdot \xi)_{t=0} = \xi \circ \operatorname{ad} X.$$

We use the same letter X for an element of $\mathscr{G}$, for the field of 1-forms on $\mathscr{G}^*$ and for the linear function on $\mathscr{G}^*$

$$X : \mathscr{G}^* \to \mathbb{R} : \xi \to \langle X, \xi \rangle.$$

A Poisson structure on $\mathscr{G}^*$ is defined by the antisymmetric 2-contravariant tensor field Λ:

$$\Lambda_\xi(X, Y) = \langle \xi, [X, Y] \rangle \qquad \forall X, Y \in \mathscr{G}.$$

If W is an orbit of G in $\mathscr{G}^*$ and if $\xi \in W$, Λ_ξ restricted to W^*_ξ is an antisymmetric, non singular, bilinear form; the 2-form F_ξ on W which corresponds to Λ_ξ by duality is non singular and defines on W a symplectic structure [2] which reads

$$F_\xi(X^*, Y^*) = \langle \xi, [X, Y] \rangle \qquad \forall X, Y \in \mathscr{G}.$$

The action of G on W is symplectic and has a moment J which is simply the canonical injection of W in $\mathscr{G}^*$; the function on W corresponding to the fundamental vector field X^* is the restriction to W of the function X:

$$i(X^*)F = -\mathrm{d}X|_W.$$

The Poisson structure Λ on $\mathscr{G}^*$ allows one to define the Poisson bracket of $\mathscr{C}^\infty$ functions on $\mathscr{G}^*$:

$$\{u, v\} \underset{\text{def}}{=} \Lambda(\mathrm{d}u, \mathrm{d}v) \qquad u, v \in \mathscr{C}^\infty(\mathscr{G}^*).$$

The symmetric algebra on $\mathscr{G}^{*\mathbb{C}}$, denoted $S(\mathscr{G})$, is naturally identified to the algebra of polynomial functions on $\mathscr{G}^*$, with complex values. For the elements $X, Y \in S(\mathscr{G})$:

$$\{X, Y\}_\xi = [X, Y]_\xi .$$

Thus the Poisson bracket on $S(\mathscr{G})$ is the extension (as derivation) of the Lie bracket of $\mathscr{G}$.

DEFINITION 1. [1] Let N be an associative subalgebra of $\mathscr{C}^\infty(\mathscr{G}^*)$ stable by the Poisson bracket; let $E(N;\nu)$ be the space of formal power series in ν ($\in \mathbb{C}$) with coefficients in N. A *deformation* of N is a bilinear map $N \times N \to E(N;\nu)$:

$$(u, v) \to u * v = \sum_{r \geqslant 0} \nu^r C_r(u, v)$$

such that:

$$C_0(u, v) = uv \qquad \frac{1}{2}(C_1(u, v) - C_1(v, u)) = \{u, v\}$$

and such that, extended to $E(N, \nu)$, it satisfies the associativity relation:

$$(u * v) * w = u * (v * w) \qquad \forall u, v, w \in N.$$

Remark. We shall also consider, in what follows, a deformation of an associative and Lie subalgebra N' of the algebra of smooth functions on an open set Ω of $\mathscr{G}^*$, stable by the action of G.

DEFINITION 2. A deformation of N ($\subset \mathscr{C}^\infty(\mathscr{G}^*)$) will be called a $*$ *product* if, for all $r > 0$,

$$C_r(u, v) = (-1)^r C_r(v, u) \qquad \forall u, v \in N.$$

DEFINITION 3. A deformation (resp. a $*$ product) on $S(\mathscr{G})$ will be called *global* if it is the restriction to $S(\mathscr{G})$ of a deformation (resp. of a $*$ product) on $\mathscr{C}^\infty(\mathscr{G}^*)$. A sufficient condition for a deformation (resp. a $*$ product) on $S(\mathscr{G})$ to be global is that the C_r be bidifferential operators; in this case we shall say that the deformation (resp. the $*$ product) is *differential*.

DEFINITION 4. A deformation of $S(\mathscr{G})$ is called *covariant*, if for all X and Y in $\mathscr{G}$:

$$\frac{1}{2\nu}(X * Y - Y * X) = \{X, Y\}.$$

Let S^p be the space of homogeneous polynomials of degree p; a deformation of

$S(\mathcal{G})$ is called *graded* if

$$\forall r, p, q \in \mathbb{N} \qquad \forall (P, Q) \in S^p \times S^q \qquad C_r(P, Q) \in S^{p+q-r}.$$

A deformation of $S(\mathcal{G})$ is said to *vanish on the constants* if for all $P \in S(\mathcal{G})$ and for all $r > 0$

$$C_r(1, P) = C_r(P, 1) = 0.$$

A $*$ product on $S(\mathcal{G})$ which is graded is necessarily covariant.

DEFINITION 5. [9] Let Ω be a G-invariant open set in $\mathcal{G}^*$; let N be an associative subalgebra of $\mathscr{C}^\infty(\Omega)$ stable by the Poisson bracket. A deformation of N is called *tangential* if for every orbit W of G, contained in Ω, and for all pairs of functions u, v in N, such that $u|_W = v|_W$ one has:

$$(u * f)|_W = (v * f)|_W \qquad \forall f \in N.$$

Let $N^G = \{ f \in N \mid f(g \cdot \xi) = f(\xi),\ \forall g \in G,\ \forall \xi \in \mathcal{G}^* \}$ be the space of G invariant functions. If $*$ is a tangential deformation, if $u \in N^G$ and $v \in N$, then:

$$(u * v)|_W = (uv)|_W.$$

Observe that $u|_W$ is a constant.

Example. Let $\mathscr{U}(\mathcal{G})$ be the universal envelopping algebra of $\mathcal{G}^{\mathbb{C}}$ and let $\sigma : S(\mathcal{G}) \to \mathscr{U}(\mathcal{G})$ be the linear bijection defined by symmetrisation:

$$\sigma(X_{i_1} \ldots X_{i_p}) = \frac{1}{p!} \sum_{s \in S_p} X_{i_{s(1)}} \circ \ldots \circ X_{i_{s(p)}}$$

where the X_{i_k} $(1 \leqslant k \leqslant p) \in \mathcal{G}$, where S_p is the permutation group of p elements and where $\circ$ denotes the product in $\mathscr{U}(\mathcal{G})$. If:

$$\mathscr{U}(\mathcal{G}) = \bigoplus_{l=0}^{\infty} \sigma(S^l) \underset{\text{not}}{=} \bigoplus_{l=0}^{\infty} \mathscr{U}^l$$

we shall denote by u_l the l-th component of u $(\in \mathscr{U}(\mathcal{G}))$. Define then, for $P \in S^p$ and $Q \in S^q$:

$$P * Q = \sum_{r=0}^{\infty} (2\nu)^r \sigma^{-1}[(\sigma(P) \circ \sigma(Q))_{(p+q-r)}].$$

One checks [7] that (1, 1) defines a $*$ product on $S(\mathcal{G})$ differential, graded, covariant. This $*$ product is in general not tangential.

2. ALGEBRAIC CONSTRUCTIONS

As in the previous paragraph $S(\mathscr{G})$ denotes the symmetric algebra associated to the complexification of the Lie algebra $\mathscr{G}$ of a connected Lie group G; it is identified with the algebra of polynomilas on $\mathscr{G}^*$; $I(\mathscr{G})$ or I will designate the algebra of polynomials on $\mathscr{G}^*$ invariant by the action of G; $S(\mathscr{G})$ is in an obvious way an $I(\mathscr{G})$ module and we shall assume in what follows that it is a free $I(\mathscr{G})$-module. As a consewuence of proposition 7, §11 of Bourbaki, Algebra, 2, [10] one has:

PROPOSITION 1. *When $S(\mathscr{G})$ is a free $I(\mathscr{G})$-module, there exists a basis e_i $(i \in \mathbb{N})$ of this module composed of homogeneous polynomials.*

Proof. Observe that the assumptions which ensure the validity of Bourbaki's proposition are satisfied. Indeed if $\overset{0}{I} = \sum_{\lambda>0} I^\lambda = \sum_{\lambda>0} S^\lambda \cap I$ and if $N = S(\mathscr{G})/\overset{0}{I}\, S(\mathscr{G}) \underset{\text{not}}{=} \Sigma N^\lambda$ (where N^λ is just the canonical projection of S^λ on N), then each N^λ is a $I^0 \sim \mathbb{C}$ module and thus admits a basis. This is the first assumption. As $S(\mathscr{G})$ is a free I-module, it is a consequence of Corollary 6, §7 of Bourbaki, Algebra, 2 [10] (or directly checked) that the canonical homomorphism $\overset{0}{I} \otimes_I S(\mathscr{G}) \to S(\mathscr{G}) : [(a, P)] \to aP$ is injective. This is the second assumption and thus proves proposition 1. ■

PROPOSITION 2. *If $S(\mathscr{G})$ is a free $I(\mathscr{G})$-module, there exists on $\mathscr{G}^*$ a deformation of $S(\mathscr{G})$, denoted $*$, which is graded, covariant, and which vanishes on the constants. Furthermore if there exists an open set Ω of $\mathscr{G}^*$, stable by G, and such that the only polynomials on $\mathscr{G}^*$, whose restriction to an orbit W contained in Ω is zero, belong to $I(\mathscr{G})$ then $*$ is a tangential deformation of $S(\mathscr{G})$ on Ω.*

Proof. Let $Z(\mathscr{G})$ be the center of $\mathscr{U}(\mathscr{G})$ (= the universal envelopping algebra of $\mathscr{G}$) and let $\theta : I(\mathscr{G}) \to Z(\mathscr{G})$ be an isomorphism of associative, commutative algebra such that:

i) $\forall a \in I^n = I \cap S^n$ one has $\theta(a) - \sigma(a) \in \mathscr{U}_{n-1} = \overset{n-1}{\underset{l=0}{\oplus}} \sigma(S^l) = \overset{n-1}{\underset{l=0}{\oplus}} \mathscr{U}^l$

(where $\sigma : S(\mathscr{G}) \to \mathscr{U}(\mathscr{G})$ is the symmetrisation map)

ii) $\forall X \in I^1$ one has $\theta(X) = \sigma(X)$.

Such an isomorphism always exist by a result of Duflo [11] (see Dixmier - - Algèbres enveloppantes, Theorem 10.4.5, 10.4.4., 10.4.2.).

Let then $P \in S^n(\mathscr{G})$; by proposition 1, P can be written in a unique way

$$P = \sum_i a_i\, e_i \qquad\qquad a_i \in I(\mathcal{G})$$

where each e_i (resp. a_i) is a homogeneous polynomial of degree d_i (resp. d_i') such that $\mathrm{d}_i + \mathrm{d}_i' = n$. Let

$$\lambda : S(\mathcal{G}) \to \mathcal{U}(\mathcal{G}) : P \to \sum_i \theta(a_i) \circ \sigma(e_i).$$

This is a linear bijection. Indeed if $X \in \mathcal{G}$, $\lambda(X) = \sigma(X)$ and if $P \in \overset{n}{\underset{l=0}{\oplus}} S^l \underset{\text{not}}{=} S_n$ one has:

$$\lambda(P) = \sum_i \theta(a_i) \circ \sigma(e_i) = \sum_i \sigma(a_i) \circ \sigma(e_i) \bmod \mathcal{U}_{n-1} =$$

$$= \sum_i \sigma(a_i\, e_i) \bmod \mathcal{U}_{n-1} =$$

$$= \sigma\Big(\sum_i a_i\, e_i\Big) \bmod \mathcal{U}_{n-1} =$$

$$= \sigma(P) \bmod \mathcal{U}_{n-1}.$$

Hence the conclusion by a recurrence argument as σ is a linear bijection.

If $P \in S^p$ and $Q \in S^q$ define

$$P * Q = \sum_{r=1}^{\infty} (2\nu)^r \lambda^{-1}[(\lambda(P) \circ \lambda(Q))_{(p+q-r)}]$$

where $u_{(n)}$ denotes the n-th component of the element u of $\mathcal{U}(\mathcal{G})$ in the decomposition

$$\mathcal{U}(\mathcal{G}) = \overset{\infty}{\underset{n=0}{\oplus}} \lambda(S^n).$$

By construction $*$ is graded; it is covariant by (ii). To prove associativity consider $P \in S^p$, $Q \in S^q$, $R \in S^r$:

$$(P * Q) * R = \sum_{n=0}^{\infty} (2\nu)^n \lambda^{-1}[(\lambda(P) \circ \lambda(Q))_{(p+q-n)}] * R =$$

$$= \sum_{n=0}^{\infty} (2\nu)^n \sum_{m=0}^{\infty} (2\nu)^m \lambda^{-1}[(\lambda(P) \circ \lambda(Q))_{(p+q-n)} \circ \lambda(R)]_{(p+q-n+r-m)} =$$

$$= \sum_{s=0}^{\infty} (2\nu)^s \lambda^{-1}\left[\sum_{t=0}^{s} [[\lambda(P) \circ \lambda(Q)]_{(p+q-t)} \circ \lambda(R)]_{(p+q+r-s)}\right].$$

Observe that if $t > s$, the formally identical terms would be 0 and thus:

$$(P * Q) * R = \sum_{s=0}^{\infty} (2\nu)^s \lambda^{-1}\left[\sum_{t=0}^{\infty} [[\lambda(P) \circ \lambda(Q)]_{(p+q-t)} \circ \lambda(R)]_{(p+q+r-s)}\right] =$$

$$= \sum_{s=0}^{\infty} (2\nu)^s \lambda^{-1}[\lambda(P) \circ \lambda(Q) \circ \lambda(R)]_{(p+q+r-s)}$$

which clearly proves associativity. Let us compute $C_0(P, Q)$ (we denote as always $P * Q = \sum_{r=0}^{\infty} \nu^r C_r(P, Q)$); with the same notation as above, and:

$$P = \sum_i a_i\, e_i \qquad \text{degree } a_i + \text{degree } e_i = p$$

$$a_i \in I(\mathcal{G})$$

$$Q = \sum_j b_j\, e_j \qquad \text{degree } b_j + \text{degree } e_j = q$$

$$b_j \in I(\mathcal{G})$$

we get:

$$P * Q = \sum_{r=0}^{\infty} (2\nu)^r \lambda^{-1}\left[\sum_{i,j} (\theta(a_i) \circ \sigma(e_i) \circ \theta(b_j) \circ \sigma(e_j))_{(p+q-r)}\right].$$

In particular:

$$C_0(P, Q) = \lambda^{-1}\left[\sum_{i,j} (\theta(a_i) \circ \sigma(e_i) \circ \theta(b_j) \circ \sigma(e_j))_{(p+q)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} (\sigma(a_i) \circ \sigma(e_i) \circ \sigma(b_j) \circ \sigma(e_j))_{(p+q)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} \sigma(a_i\, e_i\, b_j\, e_j)_{(p+q)}\right] = \lambda^{-1}[\lambda(P, Q)_{(p+q)}] =$$

$$= \lambda^{-1}[\lambda(P, Q)] = PQ$$

as $PQ \in S^{p+q}$. Similarly

$$\frac{1}{2}\,(C_1(P, Q) - C_1(Q, P)) = \lambda^{-1}\left[\left\{\sum_{i,j} \theta(a_i) \circ \sigma(e_i) \circ \theta(b_j) \circ \sigma(e_j) - \right.\right.$$

$$\left.\left. - \theta(b_j) \circ \sigma(e_j) \circ \theta(a_i) \circ \sigma(e_i)\right\}_{(p+q-1)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} \{\theta(a_i) \circ \theta(b_j) \circ ((\sigma(e_i) \circ \sigma(e_j) - \right.$$

$$\left. - \sigma(e_j) \circ \sigma(e_i))\}_{(p+q-1)}\right].$$

It is well known ([11] for example) that:

$$\sigma(e_i) \circ \sigma(e_j) - \sigma(e_j) \circ \sigma(e_i) = \sigma(\{e_i, e_j\})$$

up to terms which belong to $\overset{\mathrm{d}_i + \mathrm{d}_j - 2}{\underset{k=0}{\oplus}} \mathscr{U}^k$ (d_i (resp. d_j) = degree of e_i (resp. e_j)).

Hence:

$$\frac{1}{2}\,(C_1(P, Q) - C_1(Q, P)) = \lambda^{-1}\left[\sum_{i,j} ((\theta(a_i) \circ \theta(b_j) \circ \sigma(\{e_i, e_j\}))_{(p+q-1)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} (\sigma(a_i)\, \sigma(b_j) \circ \sigma(\{e_i, e_j\}))_{(p+q-1)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} \sigma(a_i\, b_j \{e_i, e_j\})_{(p+q-1)}\right] =$$

$$= \lambda^{-1}\left[\sum_{i,j} \sigma(\{a_j\, e_i, b_j\, e_j\})_{(p+q-1)}\right] =$$

$$= \lambda^{-1}[\lambda(\{P, Q\})_{(p+q-1)}] = \{P, Q\}.$$

This completes the proof that $*$ is a graded deformation of $S(\mathcal{G})$. Let now $a \in$ $\in I^n(\mathcal{G})$ and compute:

$$a * P = a * \sum_i a_i e_i = \sum_i \sum_{r=0}^{\infty} (2\nu)^r \lambda^{-1}[((\theta(a) \circ \theta(a_i) \circ \sigma(e_i))_{(n+p-r)}] =$$

$$= \sum_i \sum_{r=0}^{\infty} (2\nu)^r \lambda^{-1}[(\theta(aa_i) \circ \sigma(e_i))_{(n+p-r)}] =$$

$$= \sum_{r=0}^{\infty} (2\nu)^r \lambda^{-1}(\lambda(aP)_{(n+p-r)}) = aP.$$

In particular if we choose $a = 1$, we see that the deformation vanishes on the contants. The deformation is also clearly tangential on Ω; indeed if W is an orbit of G contained in Ω and if P, Q $(\in S(\mathcal{G}))$ are such that $P|_W = Q|_W$ we have:

$$(P - Q)|_W = 0.$$

Thus $(P - Q)$ belongs to $I(\mathcal{G})$ and by the formula above:

$$((P - Q) * R)|_W = ((P - Q) \cdot R)|_W = 0. \qquad \blacksquare$$

Examples. i) If $\mathcal{G}$ is semi-simple

$$S(\mathcal{G}) = I(\mathcal{G}) \otimes \mathcal{H}(\mathcal{G})$$

where $\mathcal{H}(\mathcal{G})$ is the G-invariant subspace of harmonic polynomials [12]. Thus $S(\mathcal{G})$ is a free $I(\mathcal{G})$-module and proposition 2 applies. Furthermore if rank $\mathcal{G} = l$, $I(\mathcal{G})$ is a polynomial algebra in l generators $\Delta_1, \ldots, \Delta_l$. Thus any polynomial P can be written in a unique way as:

$$P = \Sigma\, \Delta_1^{i_1} \ldots \Delta_l^{i_l} a_{i_1 \ldots i_l} \qquad (a_{i_1 \ldots i_l} \in \mathcal{H}(\mathcal{G})).$$

Define then a map $\theta : I(\mathcal{G}) \to Z(\mathcal{G})$ by:

$$\theta(\Delta_1^{i_1} \ldots \Delta_l^{i_l}) = \sigma(\Delta_1)^{i_1} \circ \ldots \circ \sigma(\Delta_l)^{i_l}.$$

It is an isomorphism satisfying (i, ii) above. The deformation constructed by means of θ has been studied in [7]; it is a $*$ product which is tangential on the open set of $\mathcal{G}^*$ which is the union of the regular G-orbits.

ii) If $\mathcal{G}$ is nilpotent, the map $\sigma|_{I(\mathcal{G})} : I(\mathcal{G}) \to Z(\mathcal{G})$ is again such an isomorphism [11]; if $S(\mathcal{G})$ happens to be a free $I(\mathcal{G})$-module, proposition 2 applies. Furthermore we will show in the next paragraph that there exists always an open set Ω in $\mathcal{G}^*$ on which the deformation of proposition 2 is tangential; it turns out that is a $*$ product.

iii) If $\mathscr{G}$ is the 3-dimensional Heisenberg algebra, let us choose a basis $\{p, q, e\}$ such that the only non vanishing product is $[p, q] = e$. The algebra $I(\mathscr{G})$ is the polynomial algebra generated by e; the algebra $S(\mathscr{G})$ is a free $I(\mathscr{G})$ module admitting as homogeneous basis $\{p^i q^j \mid i, j \in \mathbb{N}\}$. If we choose $\sigma|_{I(\mathscr{G})}$ as isomorphism, the $*$ product defined at the end of paragraph 1 has all properties indicated in proposition 2, and it induces on 2-dimensional orbits the usual Moyal $*$ product on $\mathbb{R}^2$ [1].

3. DEFORMATIONS ASSOCIATED TO NILPOTENT LIE ALGEBRAS

Let $\mathscr{G}$ be a real n-dimensional nilpotent Lie algebra and let $\mathscr{G}_0 \subset \mathscr{G}_1 \subset \ldots \subset \subset \mathscr{G}_i \subset \ldots \subset \mathscr{G}_n = \mathscr{G}$ be an increasing sequence of ideal of $\mathscr{G}$ such that $\dim \mathscr{G}_i = = i$. Let us choose a basis $\{X_i; i \leqslant n\}$ of $\mathscr{G}$ such that $X_i \in \mathscr{G}_i \setminus \mathscr{G}_{i-1}$. The following facts have been proven by M. Vergne [13].

(i) There exists a Zariski open set $\Omega \subset \mathscr{G}^*$, invariant by G (= the group $\mathrm{Ad}\,\hat{G}$, where $\hat{G}$ is the connected simply connected group with algebra $\mathscr{G}$), dense in $\mathscr{G}^*$, contained in the set of points ξ of $\mathscr{G}^*$, whose orbit has maximal dimension.

(ii) There exist $2k$ rational functions $p_1, \ldots, p_k, q_1, \ldots, q_k$ of the variables $X_i\,[X_i(\xi) \underset{\text{def}}{=} \langle X_i, \xi\rangle\,(i \leqslant n)]$ which are regular on Ω.

(iii) There exist $r = n - 2k$ polynomial functions $\lambda_1 \ldots \lambda_r$ in the variables $X_i\ (i \leqslant n)$.

(iv) There exists a Zariski open set $U \subset \mathbb{R}^{n-2k}$.

These elements have the following properties:

a) There is a diffeomorphism $\Omega \to U \times \mathbb{R}^{2k}$ defined by:

$$\xi \to (\lambda(\xi), p(\xi), q(\xi)).$$

b) There is a bijection between the set of orbits contained in Ω and the points of U:

$$L \to W^L = \{\xi \in \mathscr{G}^* \mid \lambda(\xi) = L\}.$$

c) Every orbit W contained in Ω admits a global Darboux chart defined by the variables $p_i, q_j\ (i, j \leqslant k)$:

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \qquad \{p_i, q_j\} = \delta_{ij}.$$

d) Every invariant rational function on $\mathscr{G}^*$ may be written in a unique way as a rational function of λ on $\mathbb{R}^{n-2k}$; Ω is defined as $\{\xi \in \mathscr{G}^* \mid \mu(\xi) \neq 0\}$ where μ is an ivanriant polynomial.

e) For $X \in \mathscr{G}$:

$$\langle X, \xi \rangle = \sum_{j=1}^{k} a_j(q, \lambda)\, p_j + a_0(q, \lambda)$$

where a_j $(j = 0, \ldots, k)$ is a polynomial function of $q_{j+1}, \ldots, q_k$ with coefficients which are rational functions of λ, whose denominator is a power of μ.

f) The Moyal $*$ product in p and q is defined on $C^\infty(\Omega)$, is differential, tangential and covariant by G.

We briefly recall the construction of Arnal and Cortet [6], essentially to introduce notations which we shall need later. The argument is an induction on the index of the chosen sequence of ideals $\mathscr{G}_i$ $(0 \leqslant i \leqslant n)$. We shall denote:

$$\Omega^{(i)}; p_j^{(i)}, q_j^{(i)}; \lambda_a^{(i)} \qquad (1 \leqslant j \leqslant k_i,\ 1 \leqslant a \leqslant i - 2k_i)$$

the elements defined in (i), (ii), (iii) above at the i-th step; furthermore Π_i : $: \mathscr{G}_i^* \to \mathscr{G}_{i-1}^*$ is the projection corresponding to the canonical injection of $\mathscr{G}_{i-1}$ in $\mathscr{G}_i$. One distinguishes two cases:

1) The case where $I(\mathscr{G}_i) \not\subset S(\mathscr{G}_{i-1})$; then one proves that $I(\mathscr{G}_{i-1}) \subset I(\mathscr{G}_i)$. Furthermore there exists an element $Z_i \in I(\mathscr{G}_i)$ such that:

$$Z_i = \alpha X_i + \beta \text{ (cf. notations above: } 0 \neq \alpha \in I(\mathscr{G}_{i-1}) \cap I(\mathscr{G}),\ \beta \in S(\mathscr{G}_{i-1})).$$

The maximal dimension, d_i, of the orbits in $\mathscr{G}_i^*$ is equal to d_{i-1}. One defines:

$$p_j^{(i)} = p_j^{(i-1)} \circ \Pi_i,\ q_j^{(i)} = q_j^{(i-1)} \circ \Pi_i \quad (1 \leqslant j \leqslant k_i,\ 2k_i = \mathrm{d}_i)$$

$$\lambda_a^{(i)} = \lambda_a^{(i-1)} \circ \Pi_i \qquad (1 \leqslant a \leqslant i - 1 - \mathrm{d}_i)$$

$$\lambda_{i-\mathrm{d}_i}^{(i)} = Z_i$$

and the open set $\Omega^{(i)} = \Pi_i^{-1}(\Omega^{(i-1)}) \cap \{\xi \in \mathscr{G}_i^* \mid \alpha(\xi) \neq 0\}$.

2) The case where $I(\mathscr{G}_i) \subset S(\mathscr{G}_{i-1})$; then one proves that $I(\mathscr{G}_i) \subset I(\mathscr{G}_{i-1})$. Furthermore there exists $\lambda_{c_i} \in I(\mathscr{G}_{i-1}) \setminus I(\mathscr{G}_i)$ such that $\{X_i, \lambda_{c_i}\} = \mu$ where $0 \neq \mu \in I(\mathscr{G}_i) \cap I(\mathscr{G})$. The maximal dimension, d_i, of the orbits in $\mathscr{G}_i^*$ is equal to $\mathrm{d}_{i-1} + 2$. One defines:

$$p_{k_i}^{(i)} = X_i \qquad q_{k_i}^{(i)} = \frac{\lambda_{c_i}}{\mu}$$

$$\varphi : \mathscr{G}_i^* \to \mathscr{G}_{i-1}^* : \xi \to \Pi_i(\exp[q_{k_i}^{(i)}(\xi)\, X_i] \cdot \xi)$$

$$\Omega^{(i)} = \varphi^{-1}(\Omega^{(i-1)}) \cap \{\xi \in \mathscr{G}_i^* \mid \mu(\xi) \neq 0\}$$

$$p_j^{(i)} = p_j^{(i-1)} \circ \varphi, \quad q_j^{(i)} = q_j^{(i-1)} \circ \varphi \qquad (1 \leqslant j < k_i)$$

$$\lambda_a^{(i)} = \lambda_a^{(i-1)} \circ \varphi \qquad (1 \leqslant a \leqslant i - 2 - \mathrm{d}_{i-1}).$$

Observe that φ is defined only if $\mu \neq 0$.

The variables λ, p, q which are thus introduced by a recurrence procedure allow us to define a Moyal product on $C^\infty(\Omega)$ or on the algebra N of rational regular functions on Ω. This $*$ product is in general not defined on $S(\mathscr{G})$.

PROPOSITION 3. *Let $\mathscr{G}$ be a nilpotent algebra and assume that there exists a $*$ product on $\mathscr{G}^*$, defined on $S(\mathscr{G})$, such that:*

$$a * P = aP \qquad \forall a \in I(\mathscr{G}), \quad \forall P \in S(\mathscr{G}).$$

Then this $*$ product is tangential on the open set Ω defined in 3(i).

Proof. Let $L \in U (\subset \mathbb{R}^{n-2k})$ (cf. 3, (iv)) and let W^L be an orbit in $\mathscr{G}^*$ defined by $\lambda(\xi) = L$ (cf. 3, (b)). Let P be an element of $S(\mathscr{G})$ vanishing on W^L. Then (cf. 3, (c))

$$P = f(\lambda, p, q)$$

where f is a polynomial in p, q with coefficients which are rational functions of λ, defined on Ω and vanishing for $\lambda = L$; that is:

$$P(\lambda, p, q) = \sum_{I,J} \frac{a_{I,J}(\lambda)}{b_{I,J}(\lambda)}\, p^I\, q^J$$

$$I = (i_1 \ldots i_k); i_r \in \mathbb{N} \quad (1 \leqslant r \leqslant k); p^I = p_1^{i_1} \ldots p_k^{i_k}$$

$$J = (j_1 \ldots j_k); j_r \in \mathbb{N} \quad (1 \leqslant r \leqslant k); q^J = q_1^{j_1} \ldots q_k^{j_k}$$

and $a_{I,J}$ (resp. $b_{I,J}$) are polynomials such that $a_{I,J}(L) = 0$ (resp. $b_{I,J}$ does not vanish on Ω). By the construction above, one sees that:

$$(p^I\, q^J)(\xi) = \frac{1}{\mathrm{d}_{I,J}(\xi)}\, P_{I,J}(\xi)$$

where $P_{I,J}(\xi) \in S(\mathscr{G})$ and $\mathrm{d}_{I,J}(\xi) \in I(\mathscr{G})$ and does not vanish on Ω. If we define:

$$R(\xi) = \prod_{I,J} (\mathrm{d}_{I,J}(\xi) \cdot b_{I,J}(\lambda(\xi)))\, P =$$

$$= \sum_{K,L} a_{K,L}(\lambda(\xi)) P_{K,L}(\xi) \prod_{(I,J) \neq (K,L)} \mathrm{d}_{I,J}(\xi) \cdot b_{I,J}(\lambda(\xi)))$$

the $*$ product of R by an arbitrary element Q of $S(\mathscr{G})$ is:

$$R * Q = \prod_{I,J} (\mathrm{d}_{I,J}(\xi)\, b_{I,J}(\lambda(\xi)))\, P * Q$$

(using the assumption and (3(d))). Developping:

$$R * Q = \sum_{K,L} a_{K,L}(\lambda(\xi)) \left(\prod_{(I,J) \neq (K,L)} \mathrm{d}_{I,J}(\xi) \cdot b_{I,J}(\lambda(\xi)) \right) P_{K,L} * Q$$

using again the assumption, 3(d), and the fact that $\mathrm{d}_{I,J} \in I(\mathscr{G})$. Thus by restricting to W^L:

$$\prod_{I,J} (\mathrm{d}_{I,J}(\xi) \cdot b_{I,J}(\lambda(\xi)))\big|_{W^L} \cdot (P * Q)\big|_{W^L} =$$

$$= \sum_{K,L} a_{K,L}(\lambda(\xi)) \prod_{(I,J) \neq (K,L)} \mathrm{d}_{I,J}(\xi) \cdot b_{I,J}(\lambda(\xi)))\big|_{W^L} (P_{K,L} * 0)\big|_{W^L} = 0$$

as $a_{I,J}(L) = 0$. The coefficient of the left hand side being different from 0, one gets

$$(P * Q)\big|_{W^L} = 0$$

and $*$ is tangential. ■

4. THE ALGEBRA $\mathscr{G}_{5,4}$: A COUNTEREXAMPLE

The algebra $\mathscr{G}_{5,4}$ is a 5 dimensional real nilpotent Lie algebra, which can be described, in a basis X_i $(1 \leqslant i \leqslant 5)$ by the only non vanishing commutators:

$$[X_5, X_3] = X_1 \qquad [X_4, X_3] = X_2 \qquad [X_5, X_4] = -2X_3.$$

If we define $\mathscr{G}_i$ = linear span of $(X_1, \ldots, X_i)$ we get an increasing chain of ideals as described in §3. Observe that if $i \leqslant 3$, the algebra $\mathscr{G}_i$ is abelian and the orbits are thus reduced to points; in particular $I(\mathscr{G}_3) = S(\mathscr{G}_3)$. Clearly $I(\mathscr{G}_4) \subset \subset I(\mathscr{G}_3)$ as X_3 is not an invariant function on $\mathscr{G}_4^*$; hence we are dealing with the second case of the recurrence procedure and $\mathrm{d}_4 = 2$. Let us choose $\lambda_{c_4} = X_3$, X_4 the so named basis element; then $\mu = X_2$ and

$$p_1^{(4)} = X_4\,; \qquad q_1^{(4)} = \frac{X_3}{X_2}$$

$$\lambda_1^{(4)} = X_1\,; \qquad \lambda_2^{(4)} = X_2.$$

On the other hand $I(\mathscr{G}_4) \subset I(\mathscr{G}_5)$ as X_1, X_2 are central; hence we are dealing with the first case of the recurrence procedure and $d_5 = 2$. Let us choose $Z = \alpha X_5 + \beta$ as invariant; one checks that a solution is

$$\alpha = X_2 \qquad \beta = -X_3^2 - X_1 X_4.$$

Then one gets:

$$p_1^{(5)} \underset{\text{not}}{=} p = X_4 ; \qquad q_1^{(5)} \underset{\text{not}}{=} q = \frac{X_3}{X_2}$$

$$\lambda_1^{(5)} \underset{\text{not}}{=} \lambda_1 = X_1 ; \qquad \lambda_2^{(5)} \underset{\text{not}}{=} \lambda_2 = X_2$$

$$\lambda_3^{(5)} \underset{\text{not}}{=} \lambda_3 = X_2 X_5 - X_3^2 - X_1 X_4.$$

Remark. The Moyal $*$ product in the variables p and q is not defined on $S(\mathscr{G})$. Indeed, let us compute $X_5 * X_5^2$ with $X_5 = \lambda_2 q^2 + \dfrac{\lambda_1}{\lambda_2} p + \dfrac{\lambda_3}{\lambda_2}$;

$$X_5 * X_5^2 = X_5^3 + \frac{\nu^2}{2!} C_2(X_5, X_5^2)$$

$$C_2(X_5, X_5^2) = 2\frac{\lambda_1^2}{\lambda_2} = 2\frac{X_1^2}{X_2}$$

which proves the point.

PROPOSITION 4. *There does not exist on $\mathscr{G}^*$ a $*$ product defined on $S(\mathscr{G})$, which is differential and tangential on Ω. Furthermore $S(\mathscr{G})$ is a free $I(\mathscr{G})$ module and thus the $*$ product constructed in proposition 2 on $S(\mathscr{G})$ is tangential.*

Proof. Let $*$ be the Moyal $*$-product on Ω; it is tangential differential but not defined on $S(\mathscr{G})$; assume $*'$ is another $*$ product which we assume to be tangential, differential and defined on $S(\mathscr{G})$. If we denote by C_r (resp. C_r') the cochains determining these two $*$ products, one has [1] that:

$$\tilde{\delta}(C_2 - C_2') = 0 \qquad C_2(u, v) - C_2(v, u) = 0 = C_2'(u, v) - C_2'(v, u)$$

where $\tilde{\delta}$ is the Hochschild coboundary operator. But a symmetric 2 cocycle is a coboundary [14] and $C_2 - C_2'$ beeing a bidifferential operator, one has

$$C_2 - C_2' = \tilde{\delta} A \tag{*}$$

where A is a differential one cochain [14]. If we write ∂_i for the operator

$\dfrac{\partial}{\partial X_i}$ $(1 \leqslant i \leqslant 5)$, the formulas above show that:

$$\partial_p = \partial_4 + \frac{X_1}{X_2}\,\partial_5 \qquad\qquad \partial_q = X_2 \partial_3 + 2X_3 \partial_5 .$$

The classical expression of the Moyal product reads for C_2:

$$2C_2(P, Q) = \frac{2X_1^2}{X_2}\,(\partial_{55} P \partial_5 Q + \partial_5 P \partial_{55} Q) + \text{terms with polynomial coefficients}$$

$(P, Q \in S(\mathscr{G}))$.

To prove the first part of the proposition it is sufficient to show that one can not find a differential operator A, such that $C_2' = C_2 - \tilde{\delta} A$ has polynomial coefficients. Observe that the Moyal product vanishes on the constants and so does the product $*'$ as it is tangential; hence A has no terms of order 0 in the derivatives; the terms of order 1 in the derivatives form a cocycle and thus do not contribute to the relation (*). Now recall that if $\tilde{\delta} A$ is known, A is entirely determined by combinatorial formulas [15] up to those terms of order one. As Moyal and $*'$ are tangential their cochains involve only the derivatives ∂_p and ∂_q, or in variables X_i, the derivatives ∂_3, ∂_4, ∂_5. In particular

$$\tilde{\delta} A(X_2 X_5 - X_3^2 - X_1 X_4, P) = 0 \qquad (**)$$

Using the combinatorial formulae just mentioned above, the differential operator A also involves only those derivatives (see also [9] for a later version). We shall thus write:

$$A = \sum_{i_3 + i_4 + i_5 \geqslant 2} A_{i_3 i_4 i_5}\, \partial_3^{i_3} \partial_4^{i_4} \partial_5^{i_5} .$$

Developping (**) we get:

$$0 = \sum_{j_3, j_4, j_5} (-2(j_3 + 1) X_3 A_{j_3+1\, j_4 j_5} - (j_3 + 1)(j_3 + 2) A_{j_3+2\, j_4 j_5} + \\ + (j_5 + 1) X_2 A_{j_3 j_4 j_5+1} - (j_4 + 1) X_1 A_{j_3 j_4+1\, j_5})\, \partial_3^{j_3} \partial_4^{j_4} \partial_5^{j_5} P.$$

Hence, this beeing valid for any P:

$$(j_5 + 1) X_2 A_{j_3 j_4 j_5+1} - (j_4 + 1) X_1 A_{j_3 j_4+1\, j_5} - 2(j_3 + 1) X_3 A_{j_3+1\, j_4 j_5} - \\ - (j_3 + 1)(j_3 + 2) A_{j_3+2\, j_4 j_5} = 0. \qquad (***)$$

As by assumption C_2' has polynomial coefficients, the only non polynomial term in $\tilde{\delta} A$ is:

$$\tilde{\delta} A_{002,001} = \frac{X_1^2}{X_2} + \text{polynomial terms.}$$

Hence using the combinatorial formulas, the corresponding non polynomial term in A is:

$$A_{003} = \frac{X_1^2}{3X_2} + \text{polynomial terms.}$$

To analyse the relations (***) we shall prove a technical lemma.

LEMMA. *If the integer $j_3 > a + b$, where a, b are non negative integers, the term $A_{j_3 j_4 j_5}$ of the differential operator A does not contain a polynomial of the form $X_1^a X_2^b X_3^c F(X_4, X_5)$.*

Proof of the lemma. The proof is by recurrence. Assume first that $a = b = 0$ and that the term $A_{j_3 j_4 j_5}$ contains a polynomial X_3^c for a certain $j_3 > 0$. If $j_3 > 1$ the relation (***) reads:

$$A_{j_3+1 j_4 j_5} = \frac{1}{j_3(j_3+1)} [-2j_3 X_3 A_{j_3 j_4 j_5} - (j_4+1) X_1 A_{j_3-1 j_4+1 j_5} + \\ + (j_5+1) X_2 A_{j_3-1 j_4 j_5+1}]$$

and this implies that A_{j_3+1, j_4, j_5} contains a polynomial X_3^{c+1}. In the case $j_3 = 1$, the same argument applies because the only non polynomial term does not involve X_3. Thus if $A_{j_3 j_4 j_5}$ ($j_3 \geqslant 1$) contains a polynomial X_3^c, $A_{j_3+1, j_4 j_5}$ contains a polynomial X_3^{c+1}. By recurrence A_{j_3+n, j_4, j_5} contains a polynomial X_3^{c+n}, in particular it does not vanish; but this contradicts the fact that A is a differential operator, hence of finite order. We now assume that, $\forall\, k < m$, there does not exist a polynomial $X_1^a X_2^b X_3^c$ in $A_{j_3 j_4 j_5}$ with $j_3 > a + b = k$. Examine the case $j_3 > a + b = m$; if it contains $X_1^a X_2^b X_3^c$, then using (***) one sees as above that A_{j_3+1, j_4, j_5} contains a term $X_1^a X_2^b X_3^{c+1}$ and iterating the argument one sees that this contradicts the fact that A has finite order. ∎

To conclude the proof of the first part of the proposition we shall now write 9 relations (***):

(002) $$-2X_3A_{102}-2A_{202}+3X_2A_{003}-X_1A_{012}=0$$

(102) $$-4X_3A_{202}-6A_{302}+3X_2A_{103}-X_1A_{112}=0$$

(012) $$-2X_3A_{112}-2A_{212}+3X_2A_{013}-2X_1A_{022}=0$$

(021) $$-2X_3A_{121}-2A_{221}+2X_2A_{022}-3X_1A_{031}=0$$

(120) $$-4X_3A_{220}-6A_{320}+X_2A_{121}-3X_1A_{130}=0$$

(210) $$-6X_3A_{310}-12A_{410}+X_2A_{211}-2X_1A_{220}=0$$

(011) $$-2X_3A_{111}-2A_{211}+2X_2A_{012}-2X_1A_{021}=0$$

(020) $$-2X_3A_{120}-2A_{220}+X_2A_{021}-2X_1A_{030}=0$$

(111) $$-4X_3A_{211}-6A_{311}+2X_2A_{112}-2X_1A_{121}=0.$$

«Solving» these equations, using the previous lemma, gives

$$\begin{array}{lll} A_{202}=\alpha X_1^2+\ldots & A_{012}=\beta X_1+\ldots & 2\alpha+\beta=1 \qquad \text{(i)} \\ A_{112}=-4\alpha X_1X_3+\ldots & A_{022}=4\alpha X_3^2+\ldots & \\ A_{121}=4\alpha X_2X_3+\ldots & A_{220}=\alpha X_2^2+\ldots & \\ A_{211}=2\alpha X_1X_2+\ldots & & \end{array}$$

and the two relations:

$$\beta-4\alpha=0 \qquad -24\alpha=0$$

which contradicts (i). Hence there does not exist A satisfying (*) and the first part of the proposition is proven.

To prove the second part it is enough to show that $S(\mathcal{G})$ is a free $I(\mathcal{G})$ module, as one can then apply proposition 2. Observe that any polynomial P can be written as

$$P=P_0+X_3P_1$$

where P_0 and P_1 are polynomials in X_3^2. As:

$$X_1=\lambda_1 \qquad X_2=\lambda_2 \qquad X_3^2=-\lambda_3+\lambda_2X_5-\lambda_1X_4$$

one has a decomposition of P:

$$P=\sum_{\substack{i,j\in N\\ \epsilon\in\{0,1\}}} P_{\epsilon,i,j}(\lambda_1,\lambda_2,\lambda_3)X_3^\epsilon X_4^i X_5^j$$

and $P_{\epsilon,i,j}$ belongs to $I(\mathcal{G})$. This decomposition is unique; hence $S(\mathcal{G})$ is a free

$I(\mathcal{G})$ module. ∎

5. «SPECIAL» NILPOTENT LIE ALGEBRAS

The algebraic construction done in §2 rests on the assumption that $S(\mathcal{G})$ is an $I(\mathcal{G})$ free module. Dixmier [11] gives the following example of a nilpotent algebra for which $S(\mathcal{G})$ is not a free $I(\mathcal{G})$ module; it is the 5 dimensional algebra with basis x, y, z, t, u and non vanishing brackets:

$$[x, y] = t \qquad [x, z] = u.$$

This algebra belongs to a class of nilpotent algebras studied by Corwin and Greenleaf [16], for which we present a construction of $*$ product, which is differential, tangential and graded.

DEFINITION 6. A nilpotent algebra $\mathcal{G}$ is called «special» if it contains an abelian ideal $\mathcal{M}$ whose codimension k equals 1/2 of the maximal dimension of the orbits of the adjoint group in $\mathcal{G}^*$.

Examples

(i) The 5-dimensional algebra defined above

(ii) The algebra of $2n \times 2n$ matrices which are upper triangular; the ideal $\mathcal{M}$ has dimension n^2 and is composed of those matrices A such that $A_{ik} = 0$ if $i > n$ or if $k < n$.

(iii) The algebra of dimension $(n + 2)$ with basis $\{p, 1, q, q^2, \ldots, q^n\}$ and with non vanishing commutators $[p, q^i] = iq^{i-1}$, $1 \leqslant i \leqslant n$.

If $\mathcal{G}$ is a n-dimensional «special» algebra, we shall choose the increasing sequence of ideals $\mathcal{G}_i$ in such a way that $\mathcal{G}_{n-k} = \mathcal{M}$. Hence for all $i \leqslant n - k$, $I(\mathcal{G}_i) = S(\mathcal{G}_i)$ and the orbits are reduced to points, furthermore if $n - k < i \leqslant n$ $I(\mathcal{G}_i) \subset I(\mathcal{G}_{i-1})$ and the maximal dimension of the orbits is $\mathrm{d}_i = 2(i + k - n)$.

PROPOSITION 5. *Let $\mathcal{G}$ be a n-dimensional «special» nilpotent Lie algebra. There exists a $*$ product on $\mathcal{G}^*$, defined on $S(\mathcal{G})$, differential, graded and tangential on the Zariski open set Ω.*

Proof. This $*$ product is constructed by induction using the increasing chain of ideals $\mathcal{G}_i$ and an adapted basis $\{X_j; 1 \leqslant j \leqslant n\}$ of $\mathcal{G}$ such that $X_i \in \mathcal{G}_i \setminus \mathcal{G}_{i-1}$.

If $i \leqslant n - k$ and if $P, Q \in S(\mathcal{G}_i)$ one defines

$$P \underset{i}{*} Q = PQ.$$

If $n - k < i \leqslant n$ and if $P, Q \in S(\mathcal{G}_i)$ one defines

$$P \underset{i}{*} Q = \sum_{l=0}^{\infty} \nu^l \sum_{m=0}^{l} \frac{(-1)^m}{m!\,(l-m)!} (D_i)^m \partial_i^{l-m} P \underset{(i-1)}{*} (D_i)^{l-m} \partial_i^m Q$$

where $\partial_i = \dfrac{\partial}{\partial X_i}$ and D_i is a derivation of $(S(\mathscr{G}_{i-1}), \underset{(i-1)}{*})$:

$$D_i(P \underset{i-1}{*} Q) = (D_i P) \underset{(i-1)}{*} Q + P \underset{(i-1)}{*} D_i Q$$

defined by induction:

$$D_i = \sum_{k=0}^{\infty} \nu^{2k} D_{2k}^{(i)}$$

$$D_0^{(i)} = \{X_i, .\}$$

and $D_{2k}^{(i)}$ is a differential operator on $\mathscr{G}_{i-1}^*$, with polynomial coefficients, vanishing on the constants and on the linear functions and sending S^p in S^{p-2k}. To give a meaning to the formulas above we extend D_i to $S(\mathscr{G}_i)$ (and thus also to $\mathscr{C}^\infty(\mathscr{G}_i^*)$) by requiring that:

$$D_i (X_i P) = (D_i P)\, X_i$$

and we extend $\underset{(i-1)}{*}$ to $S(\mathscr{G}_i)$ (and thus to $\mathscr{C}^\infty(\mathscr{G}_i)$) by requiring that

$$X_i \underset{(i-1)}{*} P = X_i P \qquad \forall P \in S(\mathscr{G}_i).$$

The $\underset{j}{*}$ product being the usual product of functions if $j \leqslant n-k$ we shall assume that $\underset{(i-1)}{*}$ is defined on $S(\mathscr{G}_{i-1})$ with all the properties mentioned in the proposition, and we shall construct by induction the derivation D_i. Observe first that $D_0^{(i)} = \{X_i, .\}$ is a derivation of $\underset{i-1}{*}$ up to order ν^2 (using Jacobi identity and the usual properties of the Poisson bracket). Let us assume that:

$$D' = \sum_{k=0}^{l} \nu^{2k} D_{2k}^{(i)}$$

is a derivation of $\underset{(i-1)}{*}$ up to order ν^{2l+2}. Let us introduce a new product:

$$P *' Q = (\exp - tD')(\exp tD' \cdot P \underset{(i-1)}{*} \exp tD' \cdot Q).$$

The expression $\exp - tD'$ has a meaning because the coefficient of each power of ν is a finite sum of terms. Furthermore this product is associative, has the right parties and vanishes on the constants; hence is a $*$ product. Now:

$$\sum_{2j+s=k} (-D_{2j}^{(i)} C_s^{(i-1)}(P, Q) + C_s^{(i-1)}(D_{2j}^{(i)} P, Q) + C_s^{(i-1)}(P, D_{2j}^{(i)} Q)) = 0$$

for all $k \leqslant 2l + 1$.

Thus:

$$\frac{\partial}{\partial t}(P *' Q) = \exp - tD' [- D' (\exp tD' P \underset{(i-1)}{*} \exp tD' Q) +$$

$$+ D' \cdot \exp tD' P \underset{i-1}{*} \exp tD' Q + \exp tD' P \underset{i-1}{*} D' \exp tD' Q] =$$

$$= \nu^{2l+2} \exp - tD_0^{(i)} \Big[\sum_{\substack{2j+s=2l+2 \\ s>0}} (-D_{2j}^{(i)} C_s^{(i-1)}(\exp tD_0^{(i)} P, \exp tD_0^{(i)} Q) +$$

$$+ C_s^{(i-1)}(D_{2j}^{(i)} \exp tD_0^{(i)} P, \exp tD_0^{(i)} Q) + C_s^{(i-1)}(\exp tD_0^{(i)} P, D_{2j}^{(i)} \exp tD_0^{(i)} Q))\Big]$$

$$+ \text{higher order terms.}$$

As $C'_k = C_k^{(i-1)}$ for all $k \leqslant 2l + 1$, C'_{2l+2} and $C_{2l+2}^{(i-1)}$ differ by a cocycle and in view of the symmetry, this cocycle is a coboundary:

$$C'_{2l+2} = C_{2l+2}^{(i-1)} - \tilde{\delta} \tilde{D}^i_{2l+2}.$$

The expression above show that, at order $2l + 2$, and at $t = 0$,

$$C'_{2l+2}(t = 0) = C_{2l+2}^{(i-1)}.$$

Let us define:

$$D_{2l+2}^{(i)} = \frac{\partial}{\partial t} \tilde{D}_{2l+2}^{(i)}\Big|_{t=0}.$$

By induction and use of the combinatorial formulas mentioned above [15] one sees that $D_{2j}^{(i)}$ sends polynomials of degree k, on polynomials of degree $k - 2j$, that it vanishes on the constants and annihilates polynomials of degree 1; the same is then true for $D_{2l+2}^{(i)}$. We can rewrite the previous relations as:

$$\tilde{D}^i_{2l+2} = tD_{2l+2}^{(i)} + 0(t^2)$$

$$C'_{2l+2} + t\tilde{\delta} D_{2l+2}^{(i)} + 0(t^2) = C_{2l+2}^{(i-1)}.$$

Define then:

$$D = D' + \nu^{2l+2} D_{2l+2}^{(i)}$$

and a new product:

$$P *'' Q = \exp - tD\,(\exp tDP \underset{(i-1)}{*} \exp tDQ) =$$
$$= P *' Q \text{ up to order } \nu^{2l+2} =$$
$$= P \underset{(i-1)}{*} Q \text{ up to order } \nu^{2l+2}.$$

We have:

$$C''_{2l+2}(P, Q) = C'_{2l+2}(P, Q) + t\tilde{\delta} D^{(i)}_{2l+2}(P, Q) + 0(t^2) =$$
$$= C^{(i-1)}_{2l+2}(P, Q) + 0(t^2)$$

and thus

$$\frac{\partial}{\partial t}(P *'' Q)\bigg|_{t=0} = 0 \text{ up to order } \nu^{2l+3}.$$

On the other hand:

$$\frac{\partial}{\partial t}(P *'' Q)\bigg|_{t=0} = -D(P \underset{(i-1)}{*} Q) + (DP \underset{i-1}{*} Q) + (P \underset{(i-1)}{*} DQ)$$

and thus to show that D is a derivation up to order ν^{2l+4}, we must show that $\frac{\partial}{\partial t}(P *'' Q)\bigg|_{t=0}$ vanishes to this order. This we shall now prove and this will ensure the existence of the derivation $D_{(i)}$ by recurrence.

We have:

$$\frac{\partial}{\partial t} C''_{2l+3}(P, Q)\bigg|_{t=0} = \frac{\partial}{\partial t} C'_{2l+3}(P, Q)\bigg|_{t=0} + \partial D^{(i)}_{2l+2}$$

where ∂ denotes the Chevalley coboundary. As $C_j = C'_j$ for $j \leqslant 2l+1$, $C'_{2l+2} = C^{(i-1)}_{2l+2} - t\tilde{\delta} D^{(i)}_{2l+2} + 0(t^2)$, one gets by standard results [17] that:

$$C'_{2l+3} = C^{(i-1)}_{2l+3} - t\,\partial D^{(i)}_{2l+2} + A(t) + 0(t^2)$$

where $A(t)$ is an antisymmetric bidifferential operator of order 1 in each argument. Then:

$$\frac{\partial}{\partial t} C''_{2l+3}(P, Q)\bigg|_{t=0} = \frac{\partial}{\partial t}(C'_{2l+3}(P, Q) + t\,\partial D^{(i)}_{2l+2}(P, Q))\bigg|_{t=0} =$$

$$= \frac{\partial}{\partial t}(C^{i-1}_{2l+3}(P, Q) + A(t)(P, Q))\bigg|_{t=0} = \frac{\partial}{\partial t}(A(t)(P, Q))\bigg|_{t=0}.$$

To compute this, it is enough to evaluate, for $k, j \leqslant i-1$,

$$X_k *' X_j = \exp - tD' (\exp tD'X_k \underset{i-1}{*} \exp tD'X_j) =$$

$$= \exp - tD' (\exp tD_0^{(i)} X_k \underset{i-1}{*} \exp tD_0^{(i)} X_j) =$$

$$= \exp - tD' (\exp tD_0^{(i)} X_k \cdot \exp tD_0^{(i)} X_j + \nu \{\exp tD_0^{(i)} X_k, \exp tD_0^{(i)} X_j\}).$$

Hence:

$$C'_{2l+3}(X_k, X_j) = (\exp - tD')_{2l+2} \exp tD_0^{(i)} \{X_k, X_j\}$$

where $(2l+2)$ denotes the component of this order in the ν-expansion.

Thus:

$$\frac{\partial}{\partial t} A(t)(X_k, X_j)\Big|_0 = \frac{\partial}{\partial t} ((\exp - tD')_{2l+2} \exp tD_0^{(i)} \{X_k, X_j\})\Big|_{t=0} +$$
$$+ \partial D_{2l+2}^{(i)}(X_k, X_j) = - D_{2l+2}^{(i)}(\{X_k, X_j\}) = 0.$$

Hence $\left.\frac{\partial}{\partial t} C''_{2l+3}\right|_{t=0} = 0$ and thus D is a derivation up to order ν^{2l+4}. Thus the derivation of $\underset{i-1}{*}$ on $S(\mathcal{G}_{i-1})$:

$$D^{(i)} = \{X_i, .\} + \sum_{k>0} \nu^{2k} D_{2k}^{(i)}$$

has terms for all $k > 0$ vanishing on the constants and on the functions X_j ($j \leqslant i-1$); furthermore $D_{2l}^{(i)} : S^p \to S^{p-2l}$.

To complete the proof we now check if the formula given above:

$$P \underset{i}{*} Q = \sum_{l=0}^{\infty} \nu^l \sum_{m=0}^{l} \frac{(-1)^m}{m!(l-m)!} (D_i)^m \partial_i^{l-m} P \underset{i-1}{*} (D_i)^{l-m} \partial_i^m Q$$

defines a $*$ product which is tangential on Ω.

By simple use of the definition we get:

$$C_0^{(i)}(P, Q) = PQ$$

$$C_1^{(i)}(P, Q) = \{P, Q\}_{(i)} = \{P, Q\}_{(i-1)} + \partial_i P D_i Q - D_i P \partial_i Q$$

where $\{\,,\}$ denotes the Poisson bracket on $S(\mathcal{G}_r)$ extended to $S(\mathcal{G})$

$$C_r^{(i)}(P, Q) = (-1)^r C_r^{(i)}(Q, P).$$

Also if $P, Q \in S(\mathcal{G}_{i-1})$ one has:

$$P \underset{i}{*} Q = P \underset{i-1}{*} Q.$$

In particular $u \underset{i}{*} v = uv$ if u and v belong to $S(\mathscr{U})$.

If $Q \in S(\mathscr{G}_{i-1})$:

$$X_i \underset{i}{*} Q = X_i\, Q + \nu\{X_i, Q\} + \nu^3 \cdot D_2^{(i)} Q + \dots$$

In particular this product is covariant as the $D_{2j}^{(i)}$ $(j > 0)$ vanish on the linear functions. The product is defined on $S(\mathscr{G}_i)$ and graded.

Let now $a \in I(\mathscr{G}_i)$ $(\subset S(\mathscr{M}))$; then:

$$a \underset{i}{*} P = \sum_{l=0}^{\infty} \nu^l \frac{(-1)^l}{l!} (D^{(i)})^l a \underset{i-1}{*} \partial_i^l P.$$

We shall prove that $D^{(i)}\, a = \{X_i, a\} = 0$ (as a is invariant); this will of course imply that the product is tangential. It is enough to prove that for any $u \in S(\mathscr{M})$, $D^{(i)}\, u = \{X_i, u\}$. To do this we proceed by induction on the degree of u; for the degrees 0 and 1 it is true by construction. Then for $X_j \in \mathscr{M}$, $u \in S^l(\mathscr{M})$

$$\begin{aligned} D^{(i)}(X_j \,.\, u) &= D^{(i)}(X_j \underset{i-1}{*} u) = \\ &= D^{(i)} X_j \underset{i-1}{*} u + X_j \underset{i-1}{*} D^i\, u = \\ &= \{X_i, X_j\} \underset{i-1}{*} u + X_j \underset{i-1}{*} \{X_i, u\} = \\ &= \{X_i, X_j\} u + X_j\{X_i, u\} = \{X_i, X_j\, u\}. \end{aligned}$$

The last point to prove is associativity. Making everything explicit we get:

$$(P \underset{i}{*} Q) \underset{i}{*} R = \sum_{l,l'=0}^{\infty} \sum_{m=0}^{l} \sum_{m'=0}^{l'} \sum_{r=0}^{m} \sum_{s=0}^{l-m} \nu^{l+l'} \frac{(-1)^{m+m'}}{m'!(l-m')!r!(m-r)! \cdot s!(l-m-s)!}$$

$$D^{(i)r+m'} \partial_i^{l'-m'+s} P \underset{i-1}{*} D^{(i)l'-m'+m-r} \partial_i^{m'+l-m-s} Q \underset{i-1}{*} D^{(i)l-m} \partial_i^m R$$

$$P \underset{i}{*} (Q \underset{i}{*} R) = \sum_{a,a'=0}^{\infty} \sum_{b=0}^{a} \sum_{b'=0}^{a'} \sum_{c=0}^{b} \sum_{d=0}^{a-b} \nu^{a+a'} \frac{(-1)^{b+b'}}{b'!(a'-b')!c!(b-c)!d!(a-b-d)!}$$

$$D^{(i)b} \partial_i^{a-b} P \underset{i-1}{*} D^{(i)d+b'} \partial_i^{c+a'-b'} Q \underset{i-1}{*} D^{(i)a-b-d+a'-b'} \partial_i^{b-c+b'} R$$

and associativity is proven by setting

$$a = l' + s + r \qquad\qquad c = m'$$

$$b = m' + r \qquad\qquad d = l' - m'$$

$$a' = l - r - s$$

$$b' = m - r$$

and checking the inequalities. This ends the proof. ∎

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